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LINEAR NON-STATIONARY STABILIZATION ALGORITHMS AND BROCKETT'S PROBLEM†

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Linear non-stationary stabilization algorithms which solve Brockett's problem in several cases are constructed.

Brockett [1] formulated the following problem: given a triad of matrices A, B, C, under what conditions does a matrix $K(t)$ exist such that the system

$$\dot{x} = Ax + BK(t)Cx, \quad x \in R^n \tag{1}$$

is asymptotically stable?

The problem of stabilizing system (1) using a constant matrix K is a classical one in automatic control theory [2, 3]. From that standpoint, Brockett's problem may be restated as follows: to what extent does the introduction of time-dependent matrices $K(t)$ broaden the possibilities of classical stabilization?

In stabilization problems for mechanical systems it is frequently necessary to consider a narrower class of stabilizing matrices $K(t)$. These matrices must be periodic and have zero mean value over a period $[0, T]$.

Consider, for example, linear approximation in the neighbourhood of the equilibrium position of a pendulum with vertically oscillating suspension point.

$$\ddot{\theta} + \alpha\dot{\theta} + (K(t) - \omega_0^2)\theta = 0 \tag{2}$$

where α and ω_0 are positive numbers. The most frequently considered functions $K(t)$ here have the form $\beta\sin\omega t$ [4] or the following form [5, 6]

$$K(t) = \begin{cases} \beta & \text{for } t \in [0, T/2) \\ -\beta & \text{for } t \in [T/2, T) \end{cases} \tag{3}$$

For such matrices $K(t)$ stabilization of the upper equilibrium position of the pendulum for large ω and small T is a well-known effect.

In what follows are propose algorithms for constructing periodic piecewise-constant functions $K(t)$ that solve Brockett's problem in a variety of cases.

In order to demonstrate the basic properties and advantages of these algorithms most simply, we will first consider Eq. (2) and prove the following result.

Proposition. 1. Suppose

$$\alpha^2 < 4(\beta - \omega_0^2) \tag{4}$$

Then, for any number $\tau > 0$, a number $T > \tau$ exists such that Eq. (2) with a function $K(t)$ of type (3) is asymptotically stable.

Hence, in particular, it follows that the upper equilibrium position of the pendulum can be stabilized with respect to low-frequency vertical oscillations of the suspension point. Naturally, the amplitude of the oscillation will then be large: $a = lT^2\beta/8$, where l is the pendulum length and β the absolute value of the acceleration divided by l.

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To prove Proposition 1, we observe that, without loss of generality, we can assume that

$$\beta - \omega_0^2 - \alpha^2/4 = 1 \tag{5}$$

(this may be ensured by a suitable transformation of the time variable).

Together with Eq. (2), consider the equivalent system

$$\dot{\theta} = \eta, \quad \dot{\eta} = -\alpha\eta - (K(t) - \omega_0^2)\theta \tag{6}$$

We will consider some properties of this system for $K(t) \equiv -\beta$ and $K(t) \equiv \beta$ that will be needed later.

System (6) with $K(t) \equiv -\beta$ has a saddle-type singular point with stable manifold $\eta = L_1\theta$ and unstable manifold $\eta = L_2\theta$, where

$$L_{1,2} = -\alpha/2 \mp \sqrt{\alpha^2/4 + (\beta + \omega_0^2)}$$

Now consider the fundamental matrix $X(t)$ of system (6) when $K(t) = \beta$ with initial data $X(0) = I$. It follows from condition (4) that the characteristic polynomial of this system has complex zeros; hence a number $T_1 > 0$ exists such that the linear operator $X(T_1)$ maps the straight line $\eta = L_2\theta$ onto the straight line $\eta = L_1\theta$. It follows from (5) that the straight line $\eta = L_2\theta$ is also mapped onto the straight line $\eta = L_1\theta$ by the operators $X(T_1 + 2\pi j)$ (where j are integers).

We will show that $2(T_1 + 2\pi j)$ can be chosen as the number T for sufficiently large j. To do this, we consider system (6) in the case when

$$K(t) = \begin{cases} -\beta & \text{for} \quad t \in [0, T/4) \text{ and } t \in [3T/4 \\ \beta & \text{for} \quad t \in [T/4, 3T/4) \end{cases} \tag{7}$$

We will prove that the points of the sphere $\{\eta^2 + \theta^2 \leqslant 1\}$, moving along trajectories of system (6), (7), will fall, when $t = T$, into a sphere of radius ½. To do this, we first note that the solutions of system (6) in the case when $K(t) \equiv -\beta$ with initial data at $t = 0$ from the sphere $\{\eta^2 + \theta^2 \leqslant 1\}$ will fall at $t = T/4$ into an ε-neighbourhood of the straight line $\eta = L_2\sigma$, where

$$\varepsilon = \nu_1 \exp(L_1 T/4)$$

(ν_1 is a certain number). In addition, these solutions satisfy the inequality

$$\eta(T/4)^2 + \theta(T/4)^2 \leqslant \nu_2 \exp(L_2 T/2) \tag{8}$$

(ν_2 is a certain number).

In the interval $(T/4, 3T/3)$ the motion takes place along trajectories of system (6) in the case $K(t) \equiv \beta$. As a result, the operator $X(T/2)$ maps the ε-neighbourhood of the straight line $\eta = L_2\theta$ into a $\nu_3\varepsilon$-neighbourhood of the straight line $\eta = L_1\theta$ (ν_3 is a certain number). It follows from inequality (8) and relation (5) that then

$$\eta(3T/4)^2 + \theta(3T/4)^2 \leqslant \nu_2 \exp(L_2 T/2) \tag{9}$$

In the interval $(3T/4, T)$ the motion takes place along trajectories of system (6) in the case $K(t) \equiv -\beta$. From the fact that the points $\theta(3T/4)$, $\eta(3T/4)$ are situated in the $\nu_3\varepsilon$-neighbourhood of the straight line $\eta = L_1\theta$ and from inequality (9) it follows that the points $\theta(T)$, $\eta(T)$ belong to an ε_1-neighbourhood of the straight line $\eta = L_1\theta$ and that

$$\eta(T)^2 + \theta(T)^2 \leqslant \nu_4 \exp[(L_1 + L_2)T/2] \tag{10}$$

where

$$\varepsilon_1 = \nu_5 \nu_3 \nu_1 \exp[(L_1 + L_2)T/4] \tag{11}$$

(ν_4 and ν_5 are certain numbers).

It follows from relations (10) and (11) and the inequality $L_1 + L_2 < 0$ that, by choosing T sufficiently large, we can satisfy the inequality

$$\eta(T)^2 + \theta(T)^2 \leqslant \tfrac{1}{4} \tag{12}$$

It is well-known [5] that this inequality is a sufficient condition for asymptotic stability in linear systems with periodic coefficients.

Thus, the stabilization algorithm described here is extremely simple and based on two properties of linear systems corresponding to system (6) with $K(t) \equiv -\beta$ and $K(t) \equiv \beta$. First, the solutions of the first of these systems are squeezed onto the unstable manifold more rapidly than they expand along that manifold. Second, this unstable manifold, after switching, may turn around along the trajectories of the second of these systems and, by the next switching, come to coincide with the stable manifold. Working in the interval $(3T/4, T)$, one uses the predominance of squeezing over expansion and on the whole, up to time $t = T$, entirely eliminates expansion, embedding the solutions in a sphere whose radius is as small as desired.

We will now describe an analogous algorithm for system (1).

Let us assume that a matrix K_1 exists such that the system

$$\dot{x} = (A + \mu B K_1 C)x \tag{13}$$

where μ is a scalar parameter, has a stable linear invariant manifold $L(\mu)$ for $\mu \geqslant \mu_0$, μ_0 being a certain number. We will also assume that

$$\lim_{\mu \to +\infty} L(\mu) = L_0 \tag{14}$$

and that, for any number $\delta > 0$, a number $\mu_1 \geqslant \mu_0$ exists such that

$$|x(1, x_0)| \leqslant \delta, \quad \forall x_0 \in \{|x| = 1\} \cap L(\mu), \quad \mu \geqslant \mu_1 \tag{15}$$

where $x(0, x_0) = x_0$ and the limit (14) is understood in the sense that the set $L(\mu) \cap \{|x| \leqslant 1\}$ is contained in an ε-neighbourhood of $L_0 \cap \{|x| \leqslant 1\}$, where $\varepsilon \to 0$ as $\mu \to +\infty$.

The assumption formulated here implies that the trajectories converge rapidly on the manifold $L(\mu)$ for large values of the parameter μ.

Let $M(\mu)$ denote a linear invariant manifold of system (13) such that

$$\lim_{\mu \to +\infty} M(\mu) = M_0, \quad \dim M(\mu) + \dim L(\mu) = n, \quad M(\mu) \cap L(\mu) = \{0\}$$

We will assume that $M(\mu)$ is a manifold of slow motions, that is, a number R exists such that, for all $\mu \geqslant \mu_0$,

$$|x(1, x_0)| \leqslant R, \quad \forall x_0 \in \{|x| = 1\} \cap M(\mu) \tag{16}$$

Let us assume now that a matrix K_2 exists such that, for the system

$$\dot{y} = (A + BK_2C)y \tag{17}$$

a number τ exists for which

$$Y(\tau)M_0 \subset L_0 \tag{18}$$

where $Y(t)$ is a fundamental matrix of system (17), $Y(0) = I$.

We define a $(2 + \tau)$-periodic matrix $K(t)$ as follows:

$$K(t) = \begin{cases} \mu K_1 & \text{for} \quad t \in [0, 1) \ \text{and} \ t \in [1 + \tau, 2 + \tau) \\ K_2 & \text{for} \quad t \in [1, 1 + \tau) \end{cases} \tag{19}$$

Theorem 1. System (1) with a matrix $K(t)$ of the form (19) is asymptotically stable for sufficiently large μ.

Proof. It follows from the construction of the sets L_0 and M_0 that, for any number $\varepsilon > 0$, a number $\mu_2 \geqslant \mu_0$ exists possessing the following property. For $|x_0| = 1$ and $\mu \geqslant \mu_2$, the solution $x(1, x_0)$ of system (1) lies in the ε-neighbourhood of the set

$$M_0 \cap \{|x| \leqslant R\}$$

Hence, and from condition (18), it follows that a number R_1 exists for which the following statement holds. For $|x_0| = 1, \mu \geqslant \mu$, the solution $x(1 + \tau, x_0)$ of system (1) lies in the $R_1\varepsilon$-neighbourhood of the set

$$L_0 \cap \{|x| \leqslant R_1\}$$

Hence it follows that for these solutions a number R_2 exists for which

$$|x(2 + \tau, x_0)| \leqslant R_2\varepsilon$$

Choosing ε sufficiently small (and therefore μ sufficiently large), we deduce that for all x_0 on the sphere $\{|x| = 1\}$

$$|x(2+\tau, x_0)| < 1/2$$

This implies that system (1) with a periodic matrix $K(t)$ of type (19) is indeed asymptotically stable.

In order to verify condition (18), it is frequently useful to apply the following proposition for a periodic solution $z(t)$ of the system

$$\dot{z} = Qz, \quad z \in R^n \tag{20}$$

where Q is a constant non-singular $n \times n$ matrix.

Lemma. For any vector $h \in R^n$ a number τ exists such that $h^* z(\tau) = 0$.

Proof. Supposing the contrary, we obtain $h^* z(t) \neq 0, \forall t \in R^1$. We may assume without loss of generality that $h^* z(t) > 0, \forall t \in R^1$. Hence, since $z(t)$ is periodic, we obtain

$$\lim_{t \to +\infty} Z(t) = +\infty, \quad Z(t) = \int_0^t h^* z(t) dt \tag{21}$$

On the other hand,

$$Z(t) = h^* Q^{-1}(z(t) - z(0))$$

Hence, since $z(t)$ is periodic, it immediately follows that the function $Z(t)$ is periodic, contrary to relation (21). This contradiction proves the statement of the lemma.

Theorem 2. Suppose matrices K_1 and K_2 exist satisfying the following conditions:
1) the matrix BK_1C has $n - 1$ eigenvalues with negative real parts, and $\det BK_1C = 0$,
2) for any vector $u \neq 0$ such that $BK_1Cu = 0$ and for some number λ, the vector function

$$\exp[(A + BK_2C + \lambda I)t]u$$

is periodic.

Then a periodic matrix $K(t)$ exists such that system (1) is asymptotically stable.

Proof. It follows from condition 1 of the theorem that relations (14)–(16) hold, where L_0 is a stable linear manifold of the system $\dot{z} = BK_1Cz$, and $M_0 = \{\gamma u \,|\, \gamma \in R^1\}$.

By the lemma, it follows from condition 2 of the theorem that a number τ exists for which

$$\exp[(A + BK_2C)\tau]u \in L_0$$

Thus condition (18) is satisfied. Consequently, by Theorem 1, system (1) with a matrix of type (19) is asymptotically stable.

Theorem 2 admits of a particularly simple formulation in the two-dimensional case.

Theorem 3. Let $n = 2$ and suppose matrices K_1 and K_2 exist satisfying the following conditions:

1) $\det BK_1C = 0$ and $\mathrm{tr} BK_1C \neq 0$,

2) the matrix $A + BK_2C$ has complex eigenvalues.

Then a periodic matrix $K(t)$ of type (19) exists such that system (1) is asymptotically stable.

Proof. It follows from condition 1 of the theorem that the matrix BK_1C has a non-zero eigenvalue. If this eigenvalue is negative, condition 1 of Theorem 2 holds; if it is positive, condition 1 of Theorem 2 holds for $-K_1$.

Condition 2 of Theorem 2 follows at once from condition 2 of Theorem 3.

We will now show how to apply Theorem 3 to a case which is important for control theory, i.e. when $n = 2$, B is a column vector, C is a row vector and $K(t)$ is a scalar function.

We introduce the transfer function of system (1)

$$W(p) = C(A - pI)^{-1} B = \frac{\rho p + \gamma}{p^2 + \alpha p + \beta}$$

where p is a complex variable.

In what follows, it will be assumed that $\rho \neq 0$. Then, in the case under consideration, we may assume without loss of generality that $\rho = 1$. We will also assume that the function $W(p)$ is non-degenerate, i.e., that

$$\gamma^2 - \alpha\gamma + \beta \neq 0$$

It is well known [7] that in that case system (1) may be written in the form

$$\dot{\sigma} = \eta \quad \dot{\eta} = -\alpha\eta - \beta\sigma - K(t)(\eta + \gamma\sigma) \tag{22}$$

It can be seen that stabilization of system (22) with a constant $K(t) \equiv K_0$ is possible if and only if

$$\alpha + K_0 > 0, \quad \beta + \gamma K_0 > 0$$

A necessary and sufficient condition for a number K_0 satisfying these inequalities to exist is that either $\gamma > 0$ or $\gamma \leqslant 0$, $\alpha\gamma < \beta$.

Let us consider the case when stabilization using a constant $K(t) \equiv K_0$ is impossible:

$$\gamma \leqslant 0, \quad \alpha\gamma > \beta$$

We now apply Theorem 3. In this case the matrices K_1 and K_2 are numbers. Clearly, condition 1 of Theorem 3 is satisfied, since

$$\det BK_1C = K_1 \det BC = 0, \quad \mathrm{tr}\, BK_1C = K_1CB = -K_1 \neq 0$$

Condition 2 of Theorem 3 will be satisfied if, for some K_2, the polynomial

$$p^2 + \alpha p + \beta + K_2(p + \gamma)$$

has complex zeros. It is obvious that a necessary and sufficient condition for such a number K_2 to exist is

$$\gamma^2 - \alpha\gamma + \beta > 0 \tag{23}$$

Hence, if inequality (23) holds, a periodic function $K(t)$ exists such that system (22) is asymptotically stable.

If

$$\gamma^2 - \alpha\gamma + \beta < 0 \tag{24}$$

then the following estimate holds for system (22)

$$(\eta+\gamma\sigma)^{\cdot}=(-\gamma^2+\alpha\gamma-\beta)\sigma>0,\quad \forall\sigma>0,\quad \eta=-\gamma\sigma$$

Hence it follows that for $\sigma(0) > 0$, $\eta(0) + \gamma\sigma(0) > 0$,

$$\eta(t)+\gamma\sigma(t)>0,\quad \forall t\geqslant 0$$

But then $\dot\sigma(t) = \eta(t) > 0$, $\forall t \geqslant 0$. This implies that system (22) is unstable.

Thus, the following result holds.

Theorem 4 [8]. If inequality (23) holds, a periodic function $K(t)$ exists such that system (22) is asymptotically stable.

If inequality (24) holds, there are no functions $K(t)$ for which system (22) is asymptotically stable.

This result was obtained in [9] in another class of stabilizing functions, of the form

$$K(t)=(k_0+k_1\omega\cos\omega t),\quad \omega\gg 1$$

using the method of averaging.

We will not show that the method proposed here for constructing piecewise-constant stabilizing functions also enables one to consider Brockett's problem for the case when $n = 3$, B is a column vector, C is a row vector and $K(t)$ is a scalar function.

Let us assume that the following conditions are satisfied:

1) $CB < 0$,

2) the matrix A has two complex eigenvalues with positive real parts and one negative eigenvalue,

3) for some number k, the function

$$G(t)=C\exp[(A+kBC)t]B \tag{25}$$

has at least one zero in the interval $(-\infty, 0)$,

4) $\det(B, AB, A^2B) \neq 0$.

Theorem 5. If conditions 1–4 are satisfied, a periodic function $K(t)$ exists such that system (1) is asymptotically stable.

Proof. We will describe an algorithm for constructing the required function $K(t)$. The specific feature of this problem is that strong contraction in the phase space $R^3 = \{x\}$ may be obtained by choosing $K(t) = \eta$, $\eta \gg 1$, only in one direction, parallel to the vector B. The algorithm for constructing a stabilizing function $K(t)$ therefore has more steps here than in the proof of the previous propositions. We will consider each step in succession, observing the transformations along trajectories in R^3 of the unit sphere Ω.

1. In the set $[0, 1]$ we define $K(t)$ as follows: $K(t) = \mu$, where μ is a large parameter. The sphere Ω is "flattened" into an ellipsoid Ω_1 in an ε-neighbourhood of the plane $\{Cx = 0\}$, where $\varepsilon = \varepsilon(\mu)$ is a small number. Thus, the resulting ellipsoid $\Omega_1 = x(1, \Omega)$ has one principle semiaxis of the order of $O(\varepsilon)$ and two other principle semiaxes which depend on A, B and C.

2. Consider the interval $[1, 1-\tau]$, where τ is a zero of function (25) in the interval $(-\infty, 0)$. We define $K(t)$ in this interval as follows: $K(t) = k$. Obviously, the solution $z(t, B)$ of the system

$$\dot z=(A+kBC)z \tag{26}$$

with initial data $z(0, B) = B$ satisfies the equality

$$Cz(\tau,B)=0$$

Hence it follows that the ellipse

$$\Omega_1\cap\{Cx=0\}$$

transformed along trajectories of system (1) over $[1, 1-\tau]$, will intersect the straight line $\{\lambda B\,|\,\lambda\in R^1\}$ at time $t = 1-\tau$.

3. In the interval $(1-\tau, 2-\tau)$ we define $K(t)$ as at the first step, as follows: $K(t) = \mu$, $\mu \gg 1$.

Here the intersection of the ellipse with the straight line $\{\lambda B \mid \lambda \in R^1\}$, transformed along trajectories of system (1), is "flattened" at time $t = 2 - \tau$ into an ellipse in an ε-neighbourhood of some section of the straight line $\{\nu d, \nu \in [-1, 1]\}$ situated in the plane $\{Cx = 0\}$: $Cd = 0$.

Hence, at time $t = 2 - \tau$ the unit sphere Ω, transformed along trajectories of system (1) over the interval $(0, 2 - \tau)$, is transformed into an ellipsoid situated in an $O(\varepsilon)$-neighbourhood of the section $\{\nu d, \nu \in [-1, 1]\}$.

4. In the interval $[2 - \tau, 2 - \tau + T_1]$ we define $K(t) = 0$, where the number T_1 is defined as follows. Let $\{\lambda e \mid \lambda \in R^1\}$ denote a stable linear manifold of system (26), where $e \in R^3$. Let Ψ be the plane spanned by the vectors e and B. The existence of such a plane follows from condition 4 of Theorem 5. Define T_1 to be the first time in the set $[0, +\infty)$ at which the plane Ψ is cut by a solution $z(t, d)$ of system (26) with initial data $z(0, d) = d$. The existence of T_1 follows from condition 1 of Theorem 5.

5. In the set $(2 - \tau + T_1, 2 - \tau + T_1 + T_2]$, define $K(t) = \mu$ or $K(t) = -\mu$, $\mu \gg 1$. Here the number T_2 and the sign of $K(t)$ are chosen in such a way that the vector $z(T_1, d)$ will transform into a vector $x(T_1 + T_2, z(T_1, d))$ situated in the ε-neighbourhood of the stable manifold $\{\lambda e \mid \lambda \in R^1\}$.

6. In the set $(2 - \tau + T_1 + T_2, 2 - \tau + T_1 + T_2 + T_3]$, define $K(t) = 0$, where the number T_3 is chosen so large that

$$|x(T_3, x(T_1 + T_2, z(T_1, d)))| < 1/4 \tag{27}$$

Such a number exists provided the number $\varepsilon = \varepsilon(\mu)$ mentioned in step 5 is sufficiently small.

Since the image of the unit sphere Ω, displaced along solutions of system (1), will lie at time $t = 2 - \tau + T_1 + T_2 + T_3$ in a small neighbourhood of the vector $x(T_3, x(T_1 + T_2, z(T_1, d))$, we can state, as follows from inequality (27), that this image lies in a sphere of radius $1/2$. This is equivalent to the statement that system (1), with the $(2 - \tau + T_1 + T_2 + T_3)$-periodic function $K(t)$ just constructed, is asymptotically stable. This proves Theorem 5.

Note that conditions 1–3 of Theorem 5 may be replaced by the following, formally less restrictive, conditions:

1) $CB \neq 0$,

2) a number k_1 exists such that the matrix $A + k_1BC$ has two complex eigenvalues and one negative eigenvalue,

3) a number k_2 exists such that the function $C\exp[(A + k_2BC)t]B$ has at least one zero in the interval $(-\infty, 0)$.

Note also that Theorem 5 may be considered as an extension of Theorem 4 to the three-dimensional case.

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